# Algebraic Structures: homework #11*
# Due 17 April 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Find an explicit polynomial in $\mathbb{Q}[x]$ with the root $\alpha = \sqrt{2} + \sqrt[3]{2}$ over $\mathbb{Q}$. [This polynomial does not have to be minimal.]

2. Suppose $E/F$ is a field extension and polynomials $f, g \in F[x]$ factor in $E[x]$ as
$$f = (x - \alpha_1)(x - \alpha_2),$$
$$g = (x - \beta_1)(x - \beta_2)$$
for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in E$. Prove that the degree-4 polynomial
$$h \stackrel{\text{def}}{=} \prod_{i=1}^{2} \prod_{j=1}^{2} (x - \alpha_i - \beta_j)$$
satisfies $h \in F[x]$. [ Hint: express the coefficients of $h$ in terms of those of $f$ and $g$.]

3. Let $p$ and $q$ be distinct prime numbers. Compute $[\mathbb{Q}(\sqrt{p} + \sqrt{q}) : \mathbb{Q}]$. [ Hint: consider also extensions of $\mathbb{Q}$ by $\sqrt{p}$ and $\sqrt{q}$.]

4. Prove that the polynomial $x^3 - 2x + 3$ is irreducible in $\mathbb{Q}(i)[x]$.

5. (a) Adapt Euclid's proof of infinitude of prime numbers to show the following: in the ring $\mathbb{F}_p[x]$ there are infinitely many irreducible polynomials.
   (b) Use the preceding part to deduce existence of arbitrarily large fields of characteristic $p$ that have only finitely many elements.

   [ Unlike the usual homework policy, for this problem you must follow the suggestions in the problem statement. Proofs using other ideas will not earn credit.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw11.pdf`.