Algebraic Structures: homework #10*
Due 10 April 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose R is commutative ring with 1, and A_1, A_2, A_3 are ideals in R. Suppose that there are elements $x_{1,2}, x_{2,1}, x_{1,3}, x_{3,1}, x_{2,3}, x_{3,2}$ with $x_{i,j} \in A_i$ such that $x_{i,j} + x_{j,i} = 1$ for all $i \neq j$. Define the ring homomorphism ϕ by

$$\phi\colon R \to R/A_1 \times R/A_2 \times R/A_3,$$
$$r \mapsto (r + A_1, r + A_2, r + A_3).$$

   Show that the function ϕ is surjective by giving, for each $r_1, r_2, r_3 \in R$, an explicit formula involving the r's and the x's for an element $r' \in R$ such that $\phi(r') = (r_1 + A_1, r_2 + A_2, r_3 + A_3)$. [We wrote an analogous formula for two ideals A_1, A_2 in class.]

2. Let p be a prime number. For a polynomial $f = a_0 + \cdots + a_n x^n \in \mathbb{Z}[x]$ denote $\overline{f} = b_0 + \cdots + b_n x^n \in \mathbb{F}_p[x]$ its reduction modulo p, i.e., $b_i = a_i + p\mathbb{Z}$.

   (a) Prove that if f is monic (i.e., $a_n = 1$) and $\overline{f}$ is irreducible, then f is irreducible.

   (b) Give a non-monic example (but still with $a_n \neq 0$) such that $\overline{f}$ is irreducible, but f is reducible.

   (c) Give an example of a monic f which is irreducible, but such that $\overline{f}$ is reducible.

   For parts (b) and (c), you can choose prime p. You do not have to give examples for every p.

3. Let F be a field, and let $R \subseteq F$ be an integral domain. Let F' be the smallest field containing R. Show that F' is isomorphic to the field of fractions of R.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw10.pdf`.

4. Let $f \in \mathbb{Z}[x]$. Call a number $a \in \mathbb{Z}$ *p-nice* if $p \mid f(a)$. Suppose that every $a \in \mathbb{Z}$ is either 2-nice or 3-nice (but the alternative might depend on the number a). Prove that there either all integers are 2-nice or all integers are 3-nice (or both). [Hint: suppose there are $a_2, a_3 \in \mathbb{Z}$ such that $2 \nmid f(a_2)$ and $3 \nmid f(a_3)$, and use the Chinese remainder theorem).

5. Suppose $\alpha \in \mathbb{Q}$ is a root of a monic polynomial in $\mathbb{Z}[x]$. Show that $\alpha \in \mathbb{Z}$. [This can be done 'by hand', and does not rely on what we discussed recently.]