Algebraic Structures: homework #9*
Due 6 November 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let F be a field, and let $b_1, \dots, b_k \in F$ be k distinct elements.
   (a) Show that there exists a polynomial $f \in F[x]$ such that $f(a) = 0$ if and only if $a \in \{b_1, \dots, b_k\}$.
   (b) Show that, for any choice of $c_1, \dots, c_k \in F$, there exists a polynomial $g \in F[x]$ such that $g(b_i) = c_i$ for all $i = 1, 2, \dots, k$. [Use linear combinations of polynomials that vanish at all $b_1, \dots, b_k$ except one.]
2. Suppose that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ are ideals in a ring R. Show that $\bigcup_{j=1}^{\infty} I_j$ is also an ideal in R.
3. Show that every ideal in the ring of matrices $M_2(\mathbb{R})$ is principal. [Note that this does not mean that the ring is a PID since $M_2(\mathbb{R})$ is noncommutative.]
4. Let R be a commutative ring, and I is an ideal in R. Show that the set
$$\{r \in R : \exists n \in \mathbb{Z}_+ \ r^n \in I\}$$
is an ideal in R.
5. Prove that the ring $\mathbb{Q}(\sqrt{2})$ contains no subring isomorphic to $\mathbb{Q}(\sqrt{3})$.

*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw9.pdf`.