Algebraic Structures: homework #8*
Due 30 October 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let p, q, r be distinct prime numbers. Show that a group G of order pqr contains a normal subgroup of order either p, q, or r. [Sylow]

2. Recall that $\operatorname{Aut}(H)$ is the group of automorphisms of a group H. Denote by $\phi\colon \operatorname{Aut}(H) \to \operatorname{Aut}(H)$ the identity homomorphism. Define the group $\operatorname{Hol}(H) \stackrel{\text{def}}{=} H \rtimes_\phi \operatorname{Aut}(H)$. Show that $(h, k) \cdot h_0 \stackrel{\text{def}}{=} hk(h_0)$ defines an action of the group $\operatorname{Hol}(H)$ on H.

3. Let p be a prime number. Suppose that G is a Sylow p-subgroup of S_{p^n}. Let $\beta \in S_{p^{n+1}}$ be the permutation $\beta(x) = x + p^n \pmod{p^{n+1}}$. Thinking of G as a subgroup of $S_{p^{n+1}}$ that permutes the first p^n elements of $\{1, 2, \ldots, p^{n+1}\}$, define $G_i = \beta^i G \beta^{-i}$.

   (a) Let $\widetilde{G} \stackrel{\text{def}}{=} G_1 G_2 \cdots G_p$. Show that $\widetilde{G} \cong G_1 \times G_2 \times \cdots \times G_p$.

   (b) Show that $\langle \beta, \widetilde{G} \rangle$ is a Sylow p-subgroup in $S_{p^{n+1}}$. [Hint: show that $\langle \beta \rangle$ normalizes $\widetilde{G}$.]

   (c) (Optional; not graded) Show that $\langle \beta, \widetilde{G} \rangle \cong \widetilde{G} \rtimes \langle \beta \rangle$.

4. The letter $\mathbb{Q}$ denotes the ring of rational numbers with the usual operations.

   (a) Prove that $\mathbb{Q}$ contains infinitely many distinct subrings. [Hint: a subring is also an additive subgroup.]

   (b) Prove that $\mathbb{Q}$ contains uncountably many distinct subrings.

   [Part (b) clearly implies (a), but starting with (a) is going to be easier.]

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*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw8.pdf`.

5. Let R be a non-trivial ring with 1, and $M_n(R)$ be the ring of n-by-n matrices with entries in R. Show that the center of $M_n(R)$ contains only the diagonal matrices. [Hint: consider matrices almost all of whose entries are 0.]