Algebraic Structures: homework #7*
Due 23 October 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let G be a finite abelian group.

   (a) Let p be a prime dividing $|G|$. Prove that G contains a subgroup of order p.

   (b) Prove that G contains a subgroup of every order dividing $|G|$. [Use part (a) and induction.]

2. Let m be a natural number and consider the group S_{m^2}. Break the numbers $\{1, 2, 3, \ldots, m^2\}$ into m intervals, and, for each $i = 1, 2, \ldots, m$, denote by α_i the m-cycle that shifts the elements of the i'th interval to the right, i.e.,

$$\alpha_i \stackrel{\text{def}}{=} \big(\ (i-1)m+1\ \ (i-1)m+2\ \ \ldots\ \ im\ \big).$$

   Also, define permutation $\beta \in S_{m^2}$ by

$$\beta(x) \stackrel{\text{def}}{=} x + m$$

   where the addition is understood modulo m^2. Let $G = \langle \alpha_1, \alpha_2, \ldots, \alpha_m, \beta \rangle$. Prove that $|G| = m^{m+1}$. [Hint: What is the relation between $\alpha_i\beta$ and $\beta\alpha_j$?]

3. Show that if P, P' are distinct Sylow p-subgroups of G, then there are $x \in P$ and $y \in P'$ that do not commute.

4. Suppose A, B are normal subgroups of a group G. Prove that $AB = \{ab : a \in A, b \in B\}$ is also a normal subgroup of G.

5. Have a nice break!

*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw7.pdf`.