Algebraic Structures: homework #6*
Due 9 October 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Consider A_n as a subgroup of the symmetric group on $\mathbb{Z}_+ \stackrel{\text{def}}{=} \{1, 2, 3, \cdots\}$ in the natural way, i.e., the elements of A_n permute $\{1, 2, 3, \ldots, n\}$ and leave the other elements fixed. Let $A_\infty \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{Z}_+} A_n$.
   (a) Prove that A_∞ is subgroup of $S_{\mathbb{Z}_+}$.
   (b) Prove that A_∞ is simple.
2. Suppose x, y are 3-cycles in S_n. Show that $\langle x, y \rangle$ is isomorphic to one of $\mathbb{Z}/3\mathbb{Z}$, A_4, A_5 or $(\mathbb{Z}/3\mathbb{Z})^2$.
3. Let $n \geq 5$. Show that A_n contains no proper subgroup of index less than n. [Hint: consider the action of A_n on the cosets of the subgroup.]
4. Prove that A_{n+2} contains a subgroup isomorphic to S_n. [Remark: A_{n+1} does not contain a subgroup isomorphic to S_n by the previous exercise.]
5. (a) Give an example of an infinite group G such that $\text{Aut}(G) \cong \mathbb{Z}/2\mathbb{Z}$. Justify.
   (b) Suppose G is a nontrivial finite group that contains no elements of order 2. Show that $\text{Aut}(G) \neq 1$. [This is true for all finite groups of order > 2, but requires more work.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw6.pdf`.