Algebraic Structures: homework #4*
Due 25 September 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose K, L are both normal subgroups of a group G. Show that the $K \cap L$ is also a normal subgroup of G.

2. Let $G = \mathbb{R}/\mathbb{Z}$.

   (a) Prove that $\phi \colon G \to G$ defined by $x \mapsto 2x = x + x$ is a surjective homomorphism.

   (b) Deduce the existence of a normal subgroup N of G such that $1 \neq N \neq G$ and $G/N \cong G$. What is N?

3. Let H, K be subgroups of a group G.

   (a) Show that $H \cap K \leq G$.

   (b) Assume that $|G : H|$ and $|G : K|$ are finite.
   Show that $|G : H \cap K| \leq |G : H||G : K|$.

4. (a) Prove that the groups $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.

   (b) Prove that the groups $\mathrm{GL}_2(\mathbb{R})$ and $\mathbb{R}^3$ are not isomorphic.

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*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw4.pdf`.