Algebraic Structures: homework #3*
Due 18 September 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let p be a prime and n is a positive integer. Suppose that G is a group of order p^n. Show that G contains an element of order p. [We did the $n = 1$ case in class on Monday. Trying the $n = 2$ case is a good start.]

2. Let A be the set consisting of all elements $g \in \mathrm{GL}_2(\mathbb{R})$ satisfying $|g| < \infty$.

   (a) Is A finite?

   (b) Is A a subgroup of $\mathrm{GL}_2(\mathbb{R})$?

   Justify. [If you get stuck, try conjugating, but it is possible to solve the problem without.]

3. For purposes of this problem, a *Droste group* is a group containing a proper subgroup isomorphic to itself.

   (a) Give an example of an abelian Droste group.

   (b) Give an example of a non-abelian Droste group.

   In both cases, you should write down the isomorphism.

   [The name comes from the eponymous art effect. A cool example is at `https://www.youtube.com/watch?v=9WHdyG9mJaI`. For the math behind it, see `https://www.ams.org/notices/200304/fea-escher.pdf`]

4. For a group G let $\phi_h(g) = hgh^{-1}$ be conjugation by h. In class we showed that ϕ_h is an isomoprhism $G \to G$. Let $\mathrm{Inn}(G) = \{\phi_h : h \in G\}$.

   (a) Show that $\mathrm{Inn}(G)$ is a group under the operation of composition

*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw3.pdf`.

(b) Show that the function $\pi\colon G \to \mathrm{Inn}(G)$ given by $\pi(h) = \phi_h$ is a homomorphism.

(c) Give an example of a group G such that G and $\mathrm{Inn}(G)$ are non-isomorphic.

(d) Let n be odd. Show that $\mathrm{Inn}(D_{2n})$ is isomorphic to D_{2n}.

5. Let $H \leq G$, and $g \in G$.

   (a) Prove that if the left coset gH is equal to some right coset of H, then it is in fact equal to Hg.

   (b) Prove that if $gH = Hg$, then $g^2H = Hg^2$.