# Algebraic Structures: homework #12*
# Due 27 November 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let $f \in \mathbb{Z}[x]$. Call a number $a \in \mathbb{Z}$ *p-nice* if $p \mid f(a)$. Suppose that every $a \in \mathbb{Z}$ is either 2-nice or 3-nice (but the alternative might depend on the number $a$). Prove that there either all integers are 2-nice or all integers are 3-nice (or both). [Hint: suppose there are $a_2, a_3 \in \mathbb{Z}$ such that $2 \nmid f(a_2)$ and $3 \nmid f(a_3)$, and use the Chinese remainder theorem).

2. Find an explicit non-zero polynomial in $\mathbb{Q}[x]$ with the root $\alpha = \sqrt{2} + \sqrt[3]{2}$ over $\mathbb{Q}$. [This polynomial does not have to be minimal.]

3. (a) Adapt Euclid's proof of infinitude of prime numbers to show the following: in the ring $\mathbb{F}_p[x]$ there are infinitely many irreducible polynomials.

   (b) Use the preceding part to deduce existence of arbitrarily large fields of characteristic $p$ that have only finitely many elements.

   [ Unlike the usual homework policy, for this problem you must follow the suggestions in the problem statement. Proofs using other ideas will not earn credit.]

4. Enjoy Thanksgiving!

*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw12.pdf`.