Algebraic Structures: homework #11*
Due 20 November 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Prove that if $r \in \mathbb{Z}$ can be written in the form $r = a^2 + b^2$ for some $a, b \in \mathbb{Q}$, then r can also be written as $r = c^2 + d^2$ for some $c, d \in \mathbb{Z}$.

2. Let p be a prime number. For a polynomial $f = a_0 + \cdots + a_n x^n \in \mathbb{Z}[x]$ denote $\overline{f} = b_0 + \cdots + b_n x^n \in \mathbb{F}_p[x]$ its reduction modulo p, i.e., $b_i = a_i + p\mathbb{Z}$.

   (a) Prove that if f is monic (i.e., $a_n = 1$) and $\overline{f}$ is irreducible, then f is irreducible.

   (b) Give a non-monic example (but still with $a_n \neq 0$) such that $\overline{f}$ is irreducible, but f is reducible.

   (c) Give an example of a monic f which is irreducible, but such that $\overline{f}$ is reducible.

   For parts (b) and (c), you can choose prime p. You do not have to give examples for every p.

3. Let F be a field, and let $R \subseteq F$ be an integral domain. Let F' be the smallest subfield of F containing R. Show that F' is isomorphic to the field of fractions of R.

4. Suppose $\alpha \in \mathbb{Q}$ is a root of a monic polynomial in $\mathbb{Z}[x]$. Show that $\alpha \in \mathbb{Z}$. [This can be done 'by hand', and does not rely on what we discussed recently.]

5. Let F be a field.

   (a) Prove that the ideal (x, y) in $F[x, y]$ is maximal.

   (b) Prove that the ideal (x) in $F[x, y]$ is prime, but not maximal.

*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw11.pdf`.