Algebraic Structures: homework #10*
Due 13 November 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let R and S be commutative rings with 1 and let P be a prime ideal in R. Are the following true or false? Justify.

   (a) For every ring homomorphism $\pi\colon R \to S$, the set $\pi(P)$ is a prime ideal in S.

   (b) For every ring homomorphism $\pi\colon S \to R$, the set $\pi^{-1}(P)$ is a prime ideal in S.

2. Let D be an integer that is not a square. Recall that the norm in $\mathbb{Z}[\sqrt{D}]$ is $N(a+b\sqrt{D}) = a^2 - Db^2$.

   (a) Prove that $r \in \mathbb{Z}[\sqrt{D}]$ is a unit if and only if $N(r) \in \{\pm 1\}$.

   (b) Prove that there are infinitely many units in $\mathbb{Z}[\sqrt{2}]$. [Find three units, the others are easier.]

3. Prove that $\mathbb{Z}[\sqrt{-2}]$ is a UFD.

4. Let F be a field.

   (a) Let $f \in F[x]$ and $\alpha \in F$. Show that $f(\alpha) = 0$ holds if and only if $x - \alpha \mid f$.

   (b) Let $\alpha \in F$. Show that $x - \alpha$ is a prime element in the ring $F[x]$.

   (c) Use the preceding parts to show that a non-zero polynomial $f \in F[x]$ of degree d has at most d roots.

5. Give an example of an integral domain R and two proper subrings A and B of R such that

*This homework is from `http://www.borisbukh.org/AlgebraicStructuresFall23/hw10.pdf`.

- Both A and B are integral domains,
- The fields of fractions of A and B are not isomorphic.

Justify why your example has the requisite properties.