Algebraic Structures: homework #9*
Due 5 April 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ are ideals in a ring R. Show that $\bigcup_{j=1}^{\infty} I_j$ is also an ideal in R.

2. Give an example of a group ring RG, with $G \neq 1$, which contains no zero divisors. [Hint: note that the group G must be infinite, for RG to have no zero divisors.]

3. Let I and J be ideals in a ring R.

   (a) Show that $I + J \stackrel{\text{def}}{=} \{r + s : r \in I, s \in J\}$ is the smallest ideal of R containing both I and J.

   (b) Show that $IJ \stackrel{\text{def}}{=} \{\sum_{i=1}^{n} r_i s_i : n \in \mathbb{Z}_{\geq 0}, r_i \in I, s_i \in J\}$ is an ideal contained in $I \cap J$.

   (c) Give an example when $IJ \neq I \cap J$.

4. Prove that the ring $\mathbb{Q}(\sqrt{2})$ contains no subring isomorphic to $\mathbb{Q}(\sqrt{3})$.

5. Consider $x^2 + x + 1$ as an element of polynomial ring $R = \mathbb{F}_2[x]$. Let $I = (x^2 + x + 1)$.

   (a) Show that the quotient ring R/I has four elements.

   (b) Show that the additive group $(R/I, +)$ is isomorphic to $(\mathbb{Z}/2)^2$.

   (c) Show that the group of units $(R/I)^*$ is cyclic of order 3. Deduce that R/I is a field.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw9.pdf`.