Algebraic Structures: homework #8*
Due 29 March 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose H is a finite group whose elements satisfy $h^2 = 1$ for all $h \in H$. Show that $H \cong (\mathbb{Z}/2\mathbb{Z})^n$ for some n. [Hint: first show that H is abelian.]

2. Recall that $\mathrm{Aut}(H)$ is the group of automorphisms of a group H. Denote by $\phi\colon \mathrm{Aut}(H) \to \mathrm{Aut}(H)$ the identity homomorphism. Define the group $\mathrm{Hol}(H) \stackrel{\text{def}}{=} H \rtimes_\phi \mathrm{Aut}(H)$. Show that $(h, k) \cdot h_0 \stackrel{\text{def}}{=} hk(h_0)$ defines an action of the group $\mathrm{Hol}(H)$ on H.

3. Suppose H is a finite group of order $|H| \geq 3$. Show that the group $\mathrm{Hol}(H)$ defined in the preceding exercise is non-abelian. [Hint: if H is abelian, consider the automorphism $x \mapsto -x$, and then use the first exercise. You may use it even if you did not solve it.]

4. Let R be a commutative ring with 1. Let $x \in R$ be an element satisfying $x^n = 0$ for some n (such an x is called *nilpotent*).

   (a) Show that $1 + x$ is a unit. [Hint: try small n first, and then generalize.]

   (b) Show that $u + x$ is a unit for every unit $u \in R$.

5. Let R be a non-trivial ring, and $M_n(R)$ be the ring of n-by-n matrices with entries in R. Show that the center of $M_n(R)$ contains only the diagonal matrices. [Hint: consider matrices almost all of whose entries are 0.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw8.pdf`.