Algebraic Structures: homework #4*
Due 1 March 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Compute the last two digits of $17^{17^{17}}$. [Use of calculators/computers is allowed, but is only slightly helpful.]

2. Recall that $\mathrm{GL}_n(\mathbb{R})$ is the group of n-by-n invertible matrices, endowed with the usual matrix multiplication. Let $\mathrm{SL}_n(\mathbb{R}) \stackrel{\text{def}}{=} \{A \in \mathrm{GL}_n(\mathbb{R}) : \det(A) = 1\}$.

   (a) Let det denote the determinant. Show that det is a homomorphism from $\mathrm{GL}_n(\mathbb{R})$ to $\mathbb{R}^*$.

   (b) For odd n, let $\phi\colon\ \mathrm{GL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$ be given by $\phi(A) \stackrel{\text{def}}{=} (\det A)^{-1/n} A$. Show that ϕ is a homomorphism.

   (c) Find $\mathrm{GL}_n(\mathbb{R})/\ker\phi$. [The answer should be some group whose usual definition does not involve quotients, and a proof that this group is isomorphic to $\mathrm{GL}_n(\mathbb{R})/\ker\phi$.]

3. Suppose m and n are coprime positive integers. Let G, H be cyclic groups of order m and n respectively. Show that $G \times H$ is cyclic. [Let $g \in G$ and $h \in H$ be generators, and consider (g, h).]

4. Let G be a group. Prove that intersection of any collection of normal subgroups of G is again a normal subgroup of G. [Do not assume that the collection contains only finitely many subgroups.]

5. Suppose G is a group, which might be infinite, and H is a subgroup of G.

   (a) Show that there are as many of left cosets of H in G as there are right cosets of H in G. [Hint: use inverses.]

   (b) Show that if $[G : H] = 2$, then H is normal in G.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw4.pdf`.