Algebraic Structures: homework #3*
Due 22 February 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let G be a finite group, and let H be a nonempty subset of G that is closed under the operation in G. Show that H is a subgroup. [Hint: for $g \in G$, consider $\langle g \rangle$.]

2. Let G be a cyclic group of order $n < \infty$. For each integer a define the map

$$\begin{aligned} \sigma_a \colon G &\to G, \\ g &\mapsto g^a. \end{aligned}$$

   (a) Show that σ_a is an automorphism of G if and only if a and n are coprime.

   (b) Prove that every automorphism of G is equal to σ_a for some $a \in \mathbb{Z}$.

3. Suppose elements x, y of a group G commute, and $|x| = m$ and $|y| = n$. Show that $|xy|$ divides $\operatorname{lcm}(m, n)$.

4. Let $H \leq G$, and $g \in G$.

   (a) Prove that if the left coset gH is equal to some right coset of H, then it is in fact equal to Hg.

   (b) Prove that if $gH = Hg$, then $g^2H = Hg^2$.

5. Let G be a group, let X be the set of all of its subgroups. For $g \in G$ and $H \in X$, let $g \cdot H = gHg^{-1}$. Show that this defines an action of G on X.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw3.pdf`.