Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let $G$ be a finite group, and let $H$ be a nonempty subset of $G$ that is closed under the operation in $G$. Show that $H$ is a subgroup. [Hint: for $g \in G$, consider $\langle g \rangle$.]

2. Let $G$ be a cyclic group of order $n < \infty$. For each integer $a$ define the map

$$\sigma_a : G \to G, \quad g \mapsto g^a.$$ 

(a) Show that $\sigma_a$ is an automorphism of $G$ if and only if $a$ and $n$ are coprime.

(b) Prove that every automorphism of $G$ is equal to $\sigma_a$ for some $a \in \mathbb{Z}$.

3. Suppose elements $x, y$ of a group $G$ commute, and $|x| = m$ and $|y| = n$. Show that $|xy|$ divides $\text{lcm}(m, n)$.

4. Let $H \leq G$, and $g \in G$.

(a) Prove that if the left coset $gH$ is equal to some right coset of $H$, then it is in fact equal to $Hg$.

(b) Prove that if $gH = Hg$, then $g^2H = Hg^2$.

5. Let $G$ be a group, let $X$ be the set of all of its subgroups. For $g \in G$ and $H \in X$, let $g \cdot H = gHg^{-1}$. Show that this defines an action of $G$ on $X$.