Algebraic Structures: homework #2*
Due 15 February 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let A and B be subgroups of a group G. Show that $A \cup B$ is a subgroup of G if and only if $A \subseteq B$ or $B \subseteq A$.

2. For a nonnegative integer t, a t-cycle is a cycle of length t, i.e., an element of a symmetric group of the form $(a_1\, a_2\, \cdots\, a_t)$. Suppose σ and τ are both 3-cycles, and their product $\sigma\tau$ is a t-cycle, for some t. What are possible values of t? [Your answer must consist of a set of possible values for t, examples showing that each of these values can occur, and a proof that no other values are possible.]

3. Prove that the groups $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic.

4. Let $X = \{1, 2, 3, \dots\}$ be the set of positive integers. Let S_X be the symmetric group on X.

   (a) Show that the group S_X is infinite.

   (b) Let $G = \{\pi \in S_X : \pi(i) = i \text{ for all but finitely many } i \in X\}$. Show that G is a subgroup of S_X.

5. Suppose G and H are groups, and G is abelian. Let $\phi\colon G \to H$ be a group homomorphism. Show that the set $\phi(G) \overset{\text{def}}{=} \{\phi(g) : g \in G\}$ is an abelian subgroup of H.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw2.pdf`.