Algebraic Structures: homework #13*
Due 3 May 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. (a) Prove the Lagrange interpolation formula: for any field F, and any distinct $\alpha_1, \dots, \alpha_k \in F$, and any polynomial $f \in F[x]$ of degree less than k we have

$$f = \sum_{i=1}^{k} f(\alpha_i) \prod_{j \in [k] \setminus \{i\}} \frac{x - \alpha_j}{\alpha_i - \alpha_j}.$$

[Hint: consider the difference between the left and right sides.]

   (b) Write down an analogous formula expressing a monic polynomial of degree k via its values at $\alpha_1, \alpha_2, \dots, \alpha_k$.

2. Let $m \mid n$ be positive integers. Show the restriction of the projection map $\pi \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ to the group of units $(\mathbb{Z}/n\mathbb{Z})^*$ yields a surjective map $(\mathbb{Z}/n\mathbb{Z})^* \to (\mathbb{Z}/m\mathbb{Z})^*$.

3. Let $f \in \mathbb{Z}[x]$. Call a number $a \in \mathbb{Z}$ *p-nice* if $p \mid f(a)$. Suppose that every $a \in \mathbb{Z}$ is either 2-nice or 3-nice (but the alternative might depend on a). That there either all integers are 2-nice or all integers are 3-nice (or both). [Hint: suppose there are $a_2, a_3 \in \mathbb{Z}$ such that $2 \nmid f(a_2)$ and $3 \nmid f(a_3)$, and use the Chinese remainder theorem).

4. Suppose $\alpha \in \mathbb{Q}$ is root of a monic polynomial in $\mathbb{Z}[x]$. Show that $\alpha \in \mathbb{Z}$.

5. Prove that $x^5 + 3x + 2 \in \mathbb{Z}[x]$ is irreducible. [Hint: one approach is to think modulo 3.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw13.pdf`.