Algebraic Structures: homework #12*
Due 26 April 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose $f, g \in \mathbb{Q}[x]$ are two polynomial with rational coefficients whose product has integer coefficients. Show that the product of any coefficient of f with any coefficient of g is an integer.

2. Let F be a field, and let $R \subseteq F$ be an integral domain. Let F' be the smallest field containing R. Show that F' is isomorphic to the field of fractions of R.

3. Let $R = \mathbb{Q}[x, y]$.

   (a) Show that the ideal (x, y) is not principal.

   (b) Show that the ideal (x^2, xy, y^2) is not of the form (f, g) for any $f, g \in \mathbb{Q}[x, y]$.

4. Let F be a field, and $f \in F[x]$. Show that the quotient $F[x]/(f(x))$ is a field if and only if f is irreducible.

5. Let F be a field. Give an example of a subring R of $F[x]$ such that R is a UFD, and $R \neq F[x]$ and $R \neq 0$. Justify your example.

*This homework is from http://www.borisbukh.org/AlgebraicStructures21/hw12.pdf.