Algebraic Structures: homework #11*
Due 19 April 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let F be a field.
   (a) Let $f \in F[x]$ and $\alpha \in F$. Show that $f(\alpha) = 0$ holds if and only if $x - \alpha \mid f$.
   (b) Let $\alpha \in F$. Show that $x - \alpha$ is a prime element in the ring $F[x]$.
   (c) Use the preceding parts to show that a non-zero polynomial $f \in F[x]$ of degree d has at most d roots.

2. (a) Let G be a finite abelian group, and let $r \in G$ be an element of the largest order in G. Show that $|s| \mid |r|$ for all $s \in G$. [Hint: see homework #3.]
   (b) Let F be a field, and let G be a finite subgroup of F^*. Show that the group G is cyclic. [Hint: Use the exercise above.]

3. Prove that if $r \in \mathbb{Z}$ can be written in the form $r = a^2 + b^2$ for some $a, b \in \mathbb{Q}$, then r can also be written as $r = c^2 + d^2$ for some $c, d \in \mathbb{Z}$.

4. What are all the ways of writing $80801 = 7^2 \cdot 17 \cdot 97$ as a sum of two integer squares?

5. (a) Prove that $\mathbb{Z}[\sqrt{-2}]$ is an Euclidean domain.
   (b) Give an example of a prime number $p \in \mathbb{Z}$ that is also prime in $\mathbb{Z}[\sqrt{-2}]$.
   (c) Give an example of a prime number $p \in \mathbb{Z}$ that is not a prime in $\mathbb{Z}[\sqrt{-2}]$.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw11.pdf`.