Algebraic Structures: homework #1*
Due 8 February 2021, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. For real numbers x, y, let $x \star y$ be the fractional part of $x + y$, i.e., $x \star y \stackrel{\text{def}}{=} x + y - \lfloor x + y \rfloor$. Let G be the real interval $[0, 1)$. Show that $(G, \star)$ is an abelian group.

2. Give an example of a non-empty set X and a binary operation $\circ$ on X that is commutative, associative, but which lacks the identity element. [You must prove that your example satisfied the requisite properties.]

3. Let $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

   (a) Prove that G is a group under usual addition.

   (b) Prove that $G \setminus \{0\}$ is a group under usual multiplication. [Hint: multiply both numerator and denominator by the same quantity.]

4. Prove that if a, b are commuting elements of a group G, then $(ab)^n = a^n b^n$ for all $n \in \mathbb{Z}$. [Hint: do the case $n \geq 0$ first.]

5. An element of a square matrix A is *bombastic* if $\det A$ can be changed by changing only this element. Does there exist a 5-by-5 real matrix with exactly two bombastic elements? [The aim of this exercise is to be a review of matrices. It is unrelated to topics we have covered.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw1.pdf`.