Algebra: homework #9*
Due 7 November 2022

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `andrewid_algebra_hwnum.tex` and `andrewid_algebra_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Let V, W be finite-dimensional vector spaces over a field F.
   (a) Define $W^* \stackrel{\text{def}}{=} \hom_F(W, F)$. Prove that $\hom(W, V)$ is isomorphic to $V \otimes_F W^*$ by writing an explicit isomorphism from $V \otimes_F W^*$ to $\hom(W, V)$. [For the other direction, you will need to use $\dim V, W < \infty$.]
   (b) Let $\operatorname{tr}\colon \hom(V, V) \to F$ denote the trace. Show that under the isomorphism above we have $\operatorname{tr}(w \otimes v) = w(v)$ for $w \otimes v \in V^* \otimes_F V$.

2. Let V be a finite-dimensional vector space. Let σ be the map defined on the elementary tensors in $V^{\otimes n}$ by

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1} \otimes v_n) \stackrel{\text{def}}{=} v_n \otimes v_1 \otimes \cdots \otimes v_{n-2} \otimes v_{n-1}.$$

   (a) Show that σ extends uniquely to a linear map $P\colon V^{\otimes n} \to V^{\otimes n}$.
   (b) Let $A_1, \dots, A_n\colon V \to V$ be linear maps on a finite-dimensional vector space V. Denoting by tr the trace, show that

$$\operatorname{tr}(A_1 \otimes A_2 \otimes \cdots \otimes A_n \circ P) = \operatorname{tr}(A_n A_{n-1} \cdots A_1).$$

   Here, $A \otimes A_2 \otimes \cdots \otimes A_n$ is the linear map on $V^{\otimes n}$ that corresponds to the multilinear map $(u_1, \dots, u_n) \mapsto \big(A_1(u_1), \dots, A_n(u_n)\big)$.
   (c) Prove an analogous formula for $\operatorname{tr}(A_1 \otimes \cdots \otimes A_n)$.

*This homework is from `http://www.borisbukh.org/Algebra22/hw9.pdf`.