Algebra: homework #7*
Due 24 October 2022

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `andrewid_algebra_hwnum.tex` and `andrewid_algebra_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Let F be a field.

   (a) Let $I \subset F[x_1, \dots, x_n]$ be a monomial ideal. Call a monomial x^α *minimal in I* if x^α is in I, but no monomial dividing x^α is in I. Prove that I is generated by minimal monomials.

   (b) Let $I \subset F[x_1, \dots, x_n]$ be an arbitrary ideal. Show that it admits a Gröbner basis G such that every monomial $\mathrm{LM}(g)$, for $g \in G$, is minimal in $\mathrm{LT}(I)$.

2. Let F be a field, and $A, B \subset F$ be two finite subsets of F. Define polynomials $g = \prod_{a\in A}(x-a)$ and $h = \prod_{b\in B}(y-b)$. Let $f \in F[x,y]$ be a polynomial that vanishes at every $(a,b) \in A \times B$.

   (a) Show that if f is of degree less than $|A|$ in x and degree less than $|B|$ in y, then $f = 0$.

   (b) Show that $f \in (g,h)$ (without any assumptions on the degrees of f).

*This homework is from `http://www.borisbukh.org/Algebra22/hw7.pdf`.