Algebra: homework #6*
Due 10 October 2022

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `andrewid_algebra_hwnum.tex` and `andrewid_algebra_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Let R be a commutative ring with 1, and $b \in R$ some element therein, which is not a zero divisor. Let $D = \{b^n : n \in \mathbb{Z}_{\geq 0}\}$. Show that $R[D^{-1}] \cong R[x]/(bx-1)$. [Do not assume that R is a UFD.]

2. (a) Show that $\mathbb{Z}[\sqrt{-2}]$ is a PID.

   (b) Show that the only solutions to the equation $x^2+2=y^3$ in $\mathbb{Z}$ are $(\pm 5, 3)$.

3. (a) Let R be an integral domain. Show that a non-zero degree-d polynomial in $R[x]$ has at most d roots in R.

   (b) Let p be a prime number. Suppose that $f \in \mathbb{F}_p[x]$ is a polynomial such that the function $a \mapsto f(a)$ is a ring automorphism of $\mathbb{F}_p$. Show that either $\deg f = 1$ or $\deg f \geq p$.

*This homework is from `http://www.borisbukh.org/Algebra22/hw6.pdf`.