Algebra: homework #10*
Due 14 November 2022

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `andrewid_algebra_hwnum.tex` and `andrewid_algebra_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Vote on Tuesday (or encourage your friends to vote if you cannot do so yourself).

2. Let $m_1, \dots, m_n$ be n integers such that $\prod_{i \in S} m_i$ is not a square for any nonempty set $S \subset \{1, 2, \dots, n\}$. Show that $[\mathbb{Q}(\sqrt{m_1}, \dots, \sqrt{m_n}) : \mathbb{Q}] = 2^n$. [Hint: assume that this holds both for $n-2$ and $n-1$, and use induction.]

3. We proved that if α, β are algebraic over a field k, then so are $\alpha + \beta$ and $\alpha\beta$. This exercise provides a different route to the same result.

   (a) Show that, if α is algebraic over k if and only if it is an eigenvalue of some linear operator on a finite-dimensional vector space over k.

   (b) Show that if α and β are eigenvalues of $T \colon U \to U$ and $S \colon V \to V$ respectively, then $\alpha + \beta$ and $\alpha\beta$ are eigenvalues of $T \otimes I + I \otimes S$, and $T \otimes S$.

   (c) A number $\alpha \in \mathbb{C}$ is called an *algebraic integer* if it is a root of *monic* polynomial $f \in \mathbb{Z}[x]$. Show that the algebraic integers form a subring of $\mathbb{C}$. [Hint: tweak the argument above.]

*This homework is from `http://www.borisbukh.org/Algebra22/hw10.pdf`.