Algebra: homework #9*
Due 1 November 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. We proved that if α, β are algebraic over a field k, then so are $\alpha + \beta$ and $\alpha\beta$. This exercise provides a different route to the same result.

   (a) Show that, if α is algebraic over k if and only if it is an eigenvalue of some linear operator on a finite-dimensional vector space over k.

   (b) Show that if α and β are eigenvalues of $T\colon U \to U$ and $S\colon V \to V$ respectively, then $\alpha + \beta$ and $\alpha\beta$ are eigenvalues of $T \otimes I + I \otimes S$, and $T \otimes S$.

   (c) A number $\alpha \in \mathbb{C}$ is called an *algebraic integer* if it is a root of *monic* polynomial $f \in \mathbb{Z}[x]$. Show that the algebraic integers form a subring of $\mathbb{C}$. [Hint: tweak the argument above.]

2. Let $\alpha \in \mathbb{C}$ be a root of $p \in \mathbb{Q}[x]$, which is irreducible in $\mathbb{Q}[x]$. Define the linear transformation $T\colon \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha)$ by $T(r) = \alpha r$. What is the Jordan canonical form of T, when one regards $\mathbb{Q}(\alpha)$ as a $\mathbb{Q}$-vector space?

3. Fix a field E. For a subfield F of E, let $C(F)$ be the set of all $\alpha \in E$ that are algebraic over F. Prove that $C(C(F)) = C(F)$ for any subfield F of E.

4. [REMOVED]

*This homework is from `http://www.borisbukh.org/Algebra21/hw9.pdf`.