Algebra: homework #8*
Due 27 October 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Let F be a field. Let M be an F-vector space, and set $N = \bigwedge^2 M$.
   (a) Prove that $2 \leq \dim M < \infty$, then $\bigwedge^2 N$ is not isomorphic to $\bigwedge^4 M$.
   (b) Prove that if $\dim M$ is countable, then $\bigwedge^2 N$ is isomorphic to $\bigwedge^4 M$.
2. Let R be a commutative ring with 1. An R-module M is called *Noetherian* if every submodule is finitely generated.
   (a) Show that if M_1, M_2 are Noetherian R-modules, then so is $M_1 \oplus M_2$.
   (b) Show that if M_1, M_2 are Noetherian submodules of R-module M satisfying $M = M_1 + M_2$, then M is Noetherian.
3. Suppose $T \in \mathrm{GL}_n(\mathbb{Q})$ satisfies $T^{-1} = T + T^2$. Show that $3 \mid n$.

*This homework is from `http://www.borisbukh.org/Algebra21/hw8.pdf`.