Algebra: homework #6*
Due 11 October 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Let F be a field.

   (a) Let $I \subset F[x_1, \ldots, x_n]$ be a monomial ideal. Call a monomial x^α *minimal in* I if x^α is in I, but no monomial dividing x^α is in I. Prove that I is generated by minimal monomials.

   (b) Let $I \subset F[x_1, \ldots, x_n]$ be an arbitrary ideal. Show that it admits a Gröbner basis G such that every monomial $\mathrm{LM}(g)$, for $g \in G$, is minimal in $\mathrm{LT}(I)$.

2. Let R be an integral domain, and consider the polynomial ring $R[x_1, x_2, \ldots, x_n]$. Recall that a degree of a monomial $ax_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$ is $d_1 + \cdots + d_n$ and that a polynomial is homogeneous if all of its monomials are of the same degree.

   (a) Show that a homogeneous polynomial $f \in R[x_1, \ldots, x_n]$ satisfies $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$ for all $\lambda \in R$. Is converse true, i.e., does $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$ for all $\lambda \in R$ imply that f is homogeneous?

   (b) An ideal I in $R[x_1, \ldots, x_n]$ is a *homogeneous ideal* if whenever $p \in I$ then each homogeneous component of p is also in I. Show that an ideal is a homogeneous ideal if and only if it may be generated by homogeneous polynomials. [Use induction on degrees for the "if" direction.]

3. Is the following statement true or false? If G and G' are Gröbner bases for ideals $I, I' \subset F[x_1, \ldots, x_n]$ respectively, then GG' is a Gröbner basis for the ideal II'.

*This homework is from `http://www.borisbukh.org/Algebra21/hw6.pdf`.

4. Let F be a field, and $A, B \subset F$ be two finite subsets of F. Define polynomials $g = \prod_{a \in A}(x - a)$ and $h = \prod_{b \in B}(y - b)$. Let $f \in F[x, y]$ be a polynomial that vanishes at every $(a, b) \in A \times B$.

   (a) Show that if f is of degree less than $|A|$ in x and degree less than $|B|$ in y, then $f = 0$.

   (b) Show that $f \in (g, h)$ (without any assumptions on the degrees of f).