Algebra: homework #5*
Due 4 October 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Let R be a commutative ring with 1, and $b \in R$ some element therein, which is not a zero divisor. Let $D = \{b^n : n \in \mathbb{Z}_{\geq 0}\}$. Show that $R[D^{-1}] \cong R[x]/(bx-1)$.

2. (a) Let p be a prime ideal in a ring R, which is commutative with 1. Recall that R_p is the ring $R[D^{-1}]$ for $D = R \setminus p$. Show that the ring R_p contains a unique maximal ideal.

   (b) Show that R_p is not isomorphic to the ring of p-adic integers if $R = \mathbb{Z}$.

3. (a) Show that $\mathbb{Z}[\sqrt{-2}]$ is a PID.

   (b) Show that the only solutions to the equation $x^2+2 = y^3$ in $\mathbb{Z}$ are $(\pm 5, 3)$.

4. Let p be a prime number. Suppose that $f \in \mathbb{F}_p[x]$ is a polynomial such that the function $a \mapsto f(a)$ is a ring automorphism of $\mathbb{F}_p$. Show that either $\deg f = 1$ or $\deg f \geq p$.

*This homework is from `http://www.borisbukh.org/Algebra21/hw5.pdf`.