Algebra: homework #4*
Due 27 September 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Let F be a field.

   (a) Show that $a_0 + a_1x + a_2x^2 + \cdots \in F[[x]]$ is a unit if and only if $a_0 \neq 0$.

   (b) Show that the ring of Laurent series over F is a field.

2. Let G be a finite group, and let I be an ideal in a commutative ring R, and define $R' = R/I$. Show that $R'G$ is isomorphic to a quotient of RG by a certain ideal I'. What is I'?

3. Let p be a prime number, and let $R_i = \mathbb{Z}/p^i\mathbb{Z}$. For $i < j$, let $\pi_{i,j}\colon R_j \to R_i$ be the projection map $a + p^j\mathbb{Z} \mapsto a + p^i\mathbb{Z}$. Set

$$\mathbb{Z}_p \stackrel{\text{def}}{=} \{a \in \prod_i R_i : \pi_{ji}a_j = a_i \text{for all } i < j\}.$$

   (a) Show that $\mathbb{Z}_p$ is an integral domain. It is called the ring of p-adic integers.

   (b) Show that $\mathbb{Z}_p$ contains a unique subring isomorphic to $\mathbb{Z}$.

   (c) Prove that $p\mathbb{Z}_p$ is a unique maximal ideal of $\mathbb{Z}_p$.

4. Let R, S be commutative rings, $I \subset S$ an ideal, and $\phi\colon R \to S$ is a homomorphism.

   (a) Must $\phi^{-1}(I)$ be necessarily a prime ideal if I is prime?

   (b) Must $\phi^{-1}(I)$ be necessarily a maximal ideal if I is maximal?

*This homework is from `http://www.borisbukh.org/Algebra21/hw4.pdf`.