Algebra: homework #3*
Due 20 September 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Let $G = F(x_1, \dots, x_n)$ be the free group on generators $x_1, \dots, x_n$ (the number n is called 'rank'). Let $H = \langle g^2 : g \in G \rangle$.
   (a) Show that H is normal and that $G/H \cong (\mathbb{Z}/2\mathbb{Z})^n$.
   (b) Deduce that $F(x_1, \dots, x_n)$ and $F(x_1, \dots, x_m)$ are isomorphic if and only if $m = n$.
2. (a) Let G be a group acting on a set X and $a, b \in G$. Let $X_a, X_b \subset X$ be disjoint nonempty sets, and suppose that $a^k \cdot X_b \subset X_a$ and $b^k \cdot X_a \subset X_b$ for all $k \neq 0$. Show that the a and b generate group isomorphic to $F(a, b)$.
   (b) Show that the matrices $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ generate a subgroup of $\mathrm{GL}_2(\mathbb{R})$ isomorphic to a free group of rank 2. [Hint: look at lines $y = \pm x$.]
3. Let K, H be groups and let $\phi \colon K \to \mathrm{Aut}(H)$ be a group homomorphism. Let $G = H \rtimes_\phi K$. Identify K and H with respective subgroups in G.
   (a) Show that $G/[H, H] \cong H^{ab} \rtimes_\psi K$ for a certain $\psi \colon K \to \mathrm{Aut}(H^{ab})$ and define ψ explicitly.
   (b) Let G' be the normal closure of $[H, H] \cup [K, H]$. Show that $G/G' \cong H' \times K$ for a certain group H'. What is H'?
4. Let K and H be solvable groups. Show that any group of the form $H \rtimes_\phi K$ is solvable.

*This homework is from `http://www.borisbukh.org/Algebra21/hw3.pdf`.