Algebra: homework #12*
Due 24 November 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Suppose F'/F be a field extension. Let $f \in F[x]$ be a nonzero polynomial. Let K and K' be the splitting fields of f over F and F', respectively. Let $G = \mathrm{Gal}(K/F)$ and $G' = \mathrm{Gal}(K'/F')$. Construct an injective homomorphism from G' into G.

2. Let p be a prime number.

   (a) Let $m, n \in \mathbb{N}$. Show that $\mathbb{F}_{p^m}$ contains a subfield isomorphic to $\mathbb{F}_{p^n}$ if and only if $n \mid m$, and that if such a field exists, it is unique.

   (b) Show that $\overline{\mathbb{F}_p}$ is not a finite extension over any of its proper subfields.

3. Let p be a prime. Let K/F be a Galois extension of degree p^m. Show that there is a chain of subfields

   $$F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m = K$$

   such that $[F_i : F_{i-1}] = p$ for all i.

4. Let K/F be Galois. Suppose that the coefficients of a monic polynomial $f \in K[x]$ generate K. Let

   $$\mathcal{F} = \bigl\{\sigma f : \sigma \in \mathrm{Gal}(K/F)\bigr\}.$$

   and define polynomial $g = \prod_{f' \in \mathcal{F}} f'$.

   - Show that $g \in F[x]$.
   - Show that if g is irreducible in $F[x]$, then f is irreducible in $K[x]$.

*This homework is from `http://www.borisbukh.org/Algebra21/hw12.pdf`.