Algebra: homework #11*
Due 15 November 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Recall the definition of an algebraic integer from homework #9.

   (a) Let α be an algebraic integer such that $\alpha \in \mathbb{Q}$. Show that $\alpha \in \mathbb{Z}$.

   (b) Use part (a) to give an alternative proof that $\Phi_n \in \mathbb{Z}[x]$. [You may assume that $\Phi_n \in \mathbb{Q}[x]$.]

2. Let $m_1, \dots, m_n$ be n integers such that $\prod_{i \in S} m_i$ is not a square for any non-empty set $S \subset \{1, 2, \dots, n\}$. What abstract group is $\operatorname{Aut}(\mathbb{Q}(\sqrt{m_1}, \dots, \sqrt{m_n})/\mathbb{Q})$ isomorphic to?

3. Suppose $p \in \mathbb{Z}[x]$ is a monic irreducible polynomial, all of whose complex roots are of norm 1. Prove that p is a cyclotomic polynomial. [Hint: let $\alpha_1, \dots, \alpha_n$ be the roots, and consider the polynomial with roots $\alpha_1^m, \dots, \alpha_n^m$ for growing $m \in \mathbb{N}$.]

4. Let K/F be a Galois extension, $G = \operatorname{Gal}(K/F)$, and $\alpha \in K$. Define F-linear map $T\colon K \to K$ by $T(r) = \alpha r$.

   (a) Show that the trace of T satisfies $\operatorname{tr} T = \sum_{\sigma \in G} \sigma(\alpha)$.

   (b) Show that the determinant of T satisfies $\det T = \prod_{\sigma \in G} \sigma(\alpha)$

*This homework is from `http://www.borisbukh.org/Algebra21/hw11.pdf`.