Algebra: homework #10*
Due 8 November 2021, at 10:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. Suppose F is a field of characteristic $p > 0$ such that every finite extension of F is separable.
   (a) Prove that the Frobenius endomorphism on F is an automorphism.
   (b) Let E/F be a finite extension of F. Prove that the Frobenius endomorphism on E is an automorphism.
2. Suppose F is a field of characteristic $p > 0$, and $[E : F] = p$. Show that the following are equivalent
   (a) There is $\alpha \in E$ whose minimal polynomial over F is inseperable
   (b) The minimal polynomial of every $\alpha \in E \setminus F$ over F is inseperable.
3. Let $m_1, \ldots, m_n$ be n integers such that $\prod_{i\in S} m_i$ is not a square for any non-empty set $S \subset \{1, 2, \ldots, n\}$. Show that $[\mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_n}) : \mathbb{Q}] = 2^n$. [Hint: assume that this holds both for $n-2$ and $n-1$, and use induction.]
4. Let F be a subfield of $\overline{\mathbb{Q}}(x)$. Show that either $F/\mathbb{Q}$ or $\overline{\mathbb{Q}}(x)/F$ is an algebraic extension.

*This homework is from `http://www.borisbukh.org/Algebra21/hw10.pdf`.