Algebra: homework #1*
Due 6 September 2021

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted via Gradescope. It should be typeset, except for the pictures. Pictures and commutative diagrams do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

The problems in this problem set are not particularly abstract. Some of them relate to abstractions that we will see later, others are there to refresh some undergraduate algebra, or just for fun.

1. An element of a square matrix A is *important* if $\det A$ can be changed by changing only this element. Does there exist a 10-by-10 real matrix with exactly two important elements?

2. (a) Let $a, b, c \in \mathbb{Z}$. Show that $a + b\sqrt{2} + c\sqrt{3}$ vanishes only if $a = b = c = 0$.

   (b) (Optional) Let $a, b, c, d \in \mathbb{Z}$. Show that $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{5}$ vanishes only if $a = b = c = d = 0$.

3. Show that the number of elements of order 100 in any finite group is divisible by 40.

4. Let G be a group, and suppose $A \subseteq G$ is a nonempty finite subset satisfying $|AA| = |A|$. Show that A is a left coset of some subgroup $H \leq G$.

*This homework is from `http://www.borisbukh.org/Algebra21/hw1.pdf`.