Algebraic Methods in Combinatorics: homework #5*
Due 29 April 2019, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [1+1+1+1+1] Let p be an odd prime, let $k \in \mathbb{N}$ and suppose we have a set $S = \{(a_i, b_i) \in \mathbb{F}_p^2 : i \in [p+k]\}$ of $p+k$ points that meets every non-horizontal line.

   (a) Let $F(t,u) = \prod_{i=1}^{p+k}(t + a_i + ub_i)$. Show that $F(t,u)$ is of the form $(t^p - t)G(t,u) + (u^p - u)H(t,u)$.

   (b) Let F_0, G_0, H_0 be the homogeneous components of F, G, H of degrees $p+k, k, k$ respectively. Let $f(t) = F_0(t,1), g(t) = G_0(t,1), h(t) = H_0(t,1)$. Show that $t + b_i \mid tg + h$ for every $i \in [p+k]$.

   (c) Prove that $f(t) \mid (tg+h)(t^p g' + h')$. [Here primes denote derivatives with respect to t.]

   (d) Prove that either $k \geq (p+1)/2$ or $(tg+h)(h'g - g'h) = 0$.

   (e) Deduce that if S contains no line, then $|S| \geq (3p+1)/2$.

   You may use part (a) in part (b) even if you did not solve part (a), etc.

2. [2] Let $\mathbb{F}$ be an infinite field. Suppose $f, g \in \mathbb{F}[x,y,z]$ have degrees m and n respectively, and have no common factor. Show that the set $\{f = 0\} \cap \{g = 0\}$ contains at most $(\deg f)(\deg g)$ lines.

*This homework is from `http://www.borisbukh.org/AlgMethods19/hw5.pdf`.