Algebraic Methods in Combinatorics:

homework #4*

Due 1 April 2019, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [1] Deduce Olson’s theorem for \((\mathbb{Z}/p\mathbb{Z})^n\) from the Chevalley–Warning theorem.

2. [2] Prove that if \(A, B \subset \mathbb{F}_p\), then the set \(\{a + b : a \in A, b \in B, ab \neq 1\}\) has at least \(\min(p, |A| + |B| - 3)\) elements. Also show that this is sharp for every possible pair of sizes \(|A|, |B| \geq 2\) and \(p \geq 5\).

3. [2] Suppose \(n \geq m \geq 1\) and \(H_1, \ldots, H_m\) are hyperplanes that do not cover all the vertices of \(\{0,1\}^n\). Prove that \(|\{0,1\}^n \setminus \bigcup_i H_i| \geq 2^{n-m}\).

4. [2] Let \(p\) be a prime and \(1 \leq r < p\). Suppose \(g_1, \ldots, g_{2p-2+r}\) is a sequence of \(2p-2+r\) elements from \((\mathbb{Z}/p\mathbb{Z})^2\). Show that it must necessarily contain a subsequence of length at most \(2p-r\) whose sum is zero. (Partial credit is available for \(r = 2\) case.)

5. [2] Suppose \(g_1, \ldots, g_{3p}\) is a sequence of \(3p\) elements from \((\mathbb{Z}/p\mathbb{Z})^2\) satifying \(\sum_i g_i = 0\). Show that there is a subsequence of length \(p\) that sums to zero.

6. [1] Let \(f(n)\) be the least number \(m\) such that there exist \(m\) hyperplanes covering each point of \(\{0,1\}^n\) at least twice, except for the point 0 that is not covered at all. Find \(f(n)\).

*This homework is from [http://www.borisbukh.org/AlgMethods19/hw4.pdf](http://www.borisbukh.org/AlgMethods19/hw4.pdf)