Algebraic Methods in Combinatorics: homework #3*
Due 18 March 2019, at the start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. Let $\mathbb{F}$ be an infinite field.
   (a) [2] Suppose $f_1, \ldots, f_m$ are non-zero polynomials in variables $x_1, \ldots, x_n$ over $\mathbb{F}$. Show that there is a $c \in \mathbb{F}^n$ such that none of $f_1, f_2, \ldots, f_m$ vanish at the point c.
   (b) [1] Deduce that if $W_1, \ldots, W_\ell$ are proper subspaces of $\mathbb{F}^n$ then $\bigcup_i W_i \neq \mathbb{F}^n$.
2. [2] Suppose $L \subset \mathbb{Z}/m\mathbb{Z}$ and $k \in \mathbb{Z}/m\mathbb{Z}$ and $\alpha \geq 1$ are fixed. Show that the following are equivalent:
   (a) For each n, there are N pairs of sets (A_i, B_i) with $A_i, B_i \subset [n]$ satisfying
      - the number of pairs is $N = \Omega(n^\alpha)$, and
      - $|A_i \cap B_i| \equiv k \pmod m$ for any $A \in \mathcal{F}$, and
      - $|A_i \cap B_j| \in L \pmod m$ for any two distinct i, j.

   (b) For each r, there is a matrix M of rank at most r satisfying
      - M is an N-by-N matrix, with $N = \Omega(r^\alpha)$, and
      - every diagonal entry of M is equal to $k \pmod m$,
      - every off-diagonal entry is in $L \pmod m$.

*This homework is from `http://www.borisbukh.org/AlgMethods19/hw3.pdf`.

3. [2] For a graph G, let $R(G)$ be the minimum dimension of an orthogonal representation of G over $\mathbb{R}$. Let $\theta(G)$ be the Lovász theta. Show that $\theta(G) \leq R(G)$. [Hint: let $v_1, \ldots, v_n$ be vectors representing G in the space of dimension $R(G)$. Represent the same vertices by $v_1 \otimes v_1, \ldots, v_n \otimes v_n$ and search for an appropriate handle.]

4. (a) [2] Construct a family of $\Omega(n^3)$ subsets of $[n]$, such that the size of each is divisible by 7 and such that $|A \cap B| \in \{1, 2, 4\} \pmod 7$ for every pair of distinct sets A, B.

   (b) [I do not know an answer] Is there a construction with $\{1, 2, 4\}$ and 7 replaced by $\{1, 4, 6\}$ and 19 respectively?