Algebraic Methods in Combinatorics: homework #2*
Due 18 February 2019, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let $k \notin \{1, 3\} \pmod 6$. Suppose $\mathcal{F} \subset \binom{[n]}{k}$ is $\{0, 1, 3\}$-intersecting. Show that $|\mathcal{F}| \le cn^2$, where the constant c is independent of k. (To solve this problem you do *not* need anything that we discussed on January 30th.)

2. [2] For each odd $k \ge 3$ and for each natural number n, construct a k-uniform $\{0, 2\}$-intersecting family of subset $[n]$ of size at least $n - c_k\sqrt{n}$, where c_k depends only on k.

3. Suppose X is a finite set, $x_1, \ldots, x_n \in X$ and functions $f_1, \ldots, f_n \colon X \to \mathbb{F}_3$ satisfy

   - $f_i(x_i) = 0$,
   - $f_i(x_j) \ne 0$ for $i < j$.

   Let $m = \dim \operatorname{span}\{f_1, \ldots, f_n\}$.

   (a) [2] Show that $\binom{m+1}{2} \ge n - 1$.

   (b) [1 extra credit] Prove or disprove that for infinitely many values of n the above bound is tight.

4. (a) [2] Let X be a finite set, $k \in \mathbb{N}$. Suppose $A^{(1)}, \ldots, A^{(m)} \in X^k$ are k-tuples of distinct elements from X such that for all $r < s$ there exists $i < j$ such that $A_i^{(r)} = A_j^{(s)}$. Show that the number of k-tuples is $m \le k!$.

*This homework is from `http://www.borisbukh.org/AlgMethods19/hw2.pdf`.

(b) [1] Show that the preceding bound is tight for each k.

5. (a) [2] Let $G = (V, E)$ be a graph with vertex set $V = \binom{[n]}{3}$ and a pair of sets A, B forming an edge if $|A \cap B| \in \{0, 2\}$. We showed in class that the Shannon capacity of G satisfies $s(G) \leq n + 1$ (and it is also in the Matoušek's book). Show that in fact $s(G) \leq n$.

   (b) [extra credit] Is it possible to improve the upper bound on $s(G)$ for some $n \geq 5$?

6. [1] For a graph G let $R(G)$ be the minimum dimension of an orthogonal representation of a graph G, i.e., it is the smallest integer m such that there exists a field $\mathbb{F}$ and a map $\phi \colon V(G) \to \mathbb{F}^m$ satisfying

   - $\langle \phi(v), \phi(v) \rangle \neq 0$,
   - $\langle \phi(v), \phi(u) \rangle = 0$ for $u \not\sim v$.

   Show that $6 \leq R(C_5^{\otimes 2}) \leq 9$.