Algebraic Methods in Combinatorics: homework #5*
Due 19 November 2014, at start of class

1. [2] Let $k \geq 1$ be a fixed integer. Show that the set $S_k \stackrel{\text{def}}{=} \{x \in \mathbb{F}_p : x+1, x+2\dots, x+k$ are all squares $\}$ has size $2^{-k}p + O(\sqrt{p})$ as $p \to \infty$.

2. [2+2]

   (a) Suppose $f(x,y,z) \in \mathbb{F}_p[x,y,z]$ is a non-zero polynomial whose degrees in x, y and z are at most A, B and C respectively. Show that if $(A+1)(B+1) < n$, and $A + Bn < p$ then $f(x, x^n, (x-1)^n)$ is a non-zero polynomial.

   (b) Suppose $n \mid p-1$, and let $H = \{x : x^n = 1\}$ be the subgroup of order n of $\mathbb{F}_p^*$. Use part (a) to deduce that $|H \cap (H+1)| = O(n^{2/3})$ if $n \leq p^{3/4}$.

3. [2] Let $\mathbb{F}$ be an infinite field. Suppose $f, g \in \mathbb{F}[x,y,z]$ have degrees m and n respectively, and have no common factor. Show that the set $\{f=0\} \cap \{g=0\}$ contains at most $(\deg f)(\deg g)$ lines.

*This homework is from `http://www.borisbukh.org/AlgMethods14/hw5.pdf`.