Algebraic Methods in Combinatorics: homework #4*
Due 5 November 2014, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Give an efficient algorithm, whose input is a function $F\colon \mathbb{F}_q \to \mathbb{F}_q$, and whose output is a pair of polynomials $P_1, P_2 \in \mathbb{F}_q$ with the following property: Whenever F agrees with a polynomial P in at least $(34/100)q$ points and $\deg P(x) \le q/500$, we have $P \in \{P_1, P_2\}$. (You might assume that you have an access to an efficient algorithm to factor a single-variable polynomial into irreducible factors.)

2. [2] Suppose l_1, l_2, l_3 are three non-coplanar lines in $\mathbb{F}_q^3$ meeting at a point p, and $f(x_1, x_2, x_3)$ is a polynomial vanishing on each of l_1, l_2, l_3. Suppose $\deg f \le q-1$. Show that $\nabla f(p) = 0$. Show that the condition $\deg f \le q-1$ is necessary.

3. [2] Alice picks a set S of m points in $[n]^d$, and then repeatedly does the following: whenever three points of S lie on an axis-parallel line, all points on that lie are added to S. After a while, Alice ended with the set $[n]^d$. Show that $m \ge 3^d$.

4. [1] Call a vector space L of functions from $\mathbb{F}^n$ to $\mathbb{F}$ *d-good* if whenever $f \in L$ vanishes on $d+1$ point of a line, then f also vanishes on the whole line. Show that the dimension of any d-good vector space is at most $(d+1)^n$.

*This homework is from `http://www.borisbukh.org/AlgMethods14/hw4.pdf`.