Algebraic Methods in Combinatorics: homework #3*

Due 22 October 2014, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [1] Deduce Olson's theorem for $(\mathbb{Z}/p\mathbb{Z})^n$ from the Chevalley–Warning theorem.

2. [2] Prove that if $A, B \subset \mathbb{F}_p$, then the set $\{a + b : a \in A,\ b \in B,\ ab \neq 1\}$ has at least $\min(p, |A| + |B| - 3)$ elements. Also show that this is sharp for every possible pair of sizes $|A|, |B| \geq 2$.

3. [2] Suppose $n \geq m \geq 1$ and $H_1, \ldots, H_m$ are hyperplanes that do not cover all the vertices of $\{0,1\}^n$. Prove that $|\{0,1\}^n \setminus \bigcup_i H_i| \geq 2^{n-m}$.

4. [2] Let p be a prime and $1 \leq r < p$. Suppose $g_1, \ldots, g_{2p-2+r}$ is a sequence of $2p - 2 + r$ elements from $(\mathbb{Z}/p\mathbb{Z})^2$. Show that it must necessarily contain a subsequence of length at most $2p - r$ whose sum is zero. (Partial credit is available for $r = 2$ case.)

5. [2] Suppose $g_1, \ldots, g_{3p}$ is a sequence of $3p$ elements from $(\mathbb{Z}/p\mathbb{Z})^2$ satisfying $\sum_i g_i = 0$. Show that there is a subsequence of length p that sums to zero.

6. [1] Let $f(n)$ be the least number m such that there exist m hyperplanes covering each point of $\{0,1\}^n$ at least *twice*, except for the point 0 that is not covered at all. Find $f(n)$.

*This homework is from `http://www.borisbukh.org/AlgMethods14/hw3.pdf`.