Algebraic Methods in Combinatorics: homework #2*
Due 6 October 2014, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. Suppose X is a finite set, $x_1, \dots, x_n \in X$ and functions $f_1, \dots, f_n \colon X \to \mathbb{F}_3$ satisfy

   - $f_i(x_i) = 0$,
   - $f_i(x_j) \neq 0$ for $i < j$.

   Let $m = \dim \operatorname{span}\{f_1, \dots, f_n\}$.

   (a) [2] Show that $\binom{m+1}{2} \geq n - 1$.

   (b) [1 extra credit] Prove or disprove that for infinitely many values of n the above bound is tight.

2. (a) [2] Let X be a finite set, $k \in \mathbb{N}$. Suppose $A^{(1)}, \dots, A^{(m)} \in X^k$ are k-tuples of distinct elements from X such that for all $r < s$ there exists $i < j$ such that $A_i^{(r)} = A_j^{(s)}$. Show that the number of k-tuples is $m \leq k!$.

   (b) [1] Show that the preceding bound is tight for each k.

3. (a) [2] Let $G = (V, E)$ be a graph with vertex set $V = \binom{[n]}{3}$ and a pair of sets A, B forming an edge if $|A \cap B| \in \{0, 2\}$. We showed in class that the Shannon capacity of G satisfies $s(G) \leq n + 1$ (and it is also in the Matoušek's book). Show that in fact $s(G) \leq n$.

*This homework is from `http://www.borisbukh.org/AlgMethods14/hw2.pdf`.

(b) [extra credit; I do not know the answer] Is it possible to improve the upper bound on $s(G)$ for some $n \geq 5$?

4. [1] For a graph G let $R(G)$ be the minimum dimension of an orthogonal representation of a graph G, i.e., it is the smallest integer m such that there exists a field $\mathbb{F}$ and a map $\phi\colon V(G) \to \mathbb{F}^m$ satisfying

   - $\langle \phi(v), \phi(v) \rangle \neq 0$,

   - $\langle \phi(v), \phi(u) \rangle = 0$ for $u \not\sim v$.

   Show that $6 \leq R(C_5^{\otimes 2}) \leq 9$.

5. [2] Suppose $[n]$ is the vertex set of a graph G, and there are functions $f_1, \ldots, f_n\colon X \to \mathbb{F}$ and elements $v_1, \ldots, v_n$ that satisfy

   - $f_i(v_i) \neq 0$ for all i,
   - $f_i(v_j) = 0$ for all $i < j$ that satisfy $i \not\sim j$,

   Show that the Shannon capacity of G obeys $s(G) \leq \dim \operatorname{span}\{f_1, \ldots, f_n\}$.