Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Suppose $S$ and $T$ are two families of sets such that $|S \cap T|$ is odd for every $S \in S$ and $T \in T$. Show that $|S||T| \leq 2^{n-1}$.

2. Not far from Eventown and Oddtown there lies another settlement: Square-town. There each club has square-many members, and any two clubs share square-many members.
   
   (a) [1] Show that Squaretown can have no more than $2^{O(\sqrt{n \log n})}$ clubs.
   
   (b) [1] Show that Squaretown can have as many as $2^{\sqrt{n}}$ clubs.

3. [1] Let $v_1, \ldots, v_m$ be vectors with $n$ integer entries, each of which is 0 or 1. Show that they are linearly independent over $\mathbb{Q}$ if and only if they are linearly independent over $\mathbb{F}_p$ for all sufficiently large $p$. How large is “sufficiently large”?

4. [2] Let $s$ be a positive integer. Suppose $A_1, \ldots, A_m$ are subsets of an $n$-element set such that the sizes of $A_1, \ldots, A_m$ are not divisible by $s$, but $|A_i \cap A_j|$ are, for any distinct $i$ and $j$. For $s = 6$, show that $m \leq 2n$. For general $s$, show that there is a constant $c(s)$ such that $m \leq c(s)n$. (Open problem: must $c(s)$ depend on $s$?)

*This homework is from http://www.borisbukh.org/AlgMethods14/hw1.pdf
5. (a) [1] Show that for each $B$ there exists a number $\varepsilon > 0$ such that if $f(x)$ is a non-zero polynomial whose entries are integers not exceeding $B$, then $f(x) \neq 0$ for all $x \in (0, \varepsilon)$.

(b) [1] Show that there exists $r_0 = r_0(d)$ sufficiently large so that the following holds for all $r > r_0$. Whenever $P \subset \mathbb{R}^d$ is a two-distance set in which distance between any two points are 1 and $r$, then $|P| \leq d + 1$. (Hint: prove the same for one-distance sets first.)

6. [2] Let $A_1, \ldots, A_m$ be subsets of an $n$-element set. Suppose $|A_i|$ is odd for all $i$, and all triplewise intersections $|A_i \cap A_j \cap A_k|$ are even (for distinct $i$, $j$ and $k$). Show that there is a constant $C$ such that $m = O(n^C)$. 